Application of *k*-Laplace transform to estimate the time value of money in quantitative finance

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Abstract

In this study, the fractional Laplace transforms is shown to be more useful to deduce existing value rules. Not many analytic solutions exist for present value problems; with *k*-Laplace transforms, some of the closed form solutions are efficient. The boundary of the integral is from 0 to some finite quantity. When the change takes the upper bound to infinity, then the present value of a future cash flow is the-Laplace transform of the current cash flow.

Keywords:- *Laplace transform, finance, parameter, time variable function.*

Introduction:

The Laplace transform of *f* (*t*) (Lokenath Debnath & Dambaru Bhatta, 2015) is defined by

$$L\left\{f(t)\right\} = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{1.1}$$

where e^{-st} is called the kernel of the transform and $f(t)$ is defined for all t in $(0,\infty)$. In this study, the simplest and the most profound application of k- Laplace transform is to estimate the time value of money. The present value problem is in Finance. From *Process Dynamics: Laplace transforms, (Tham, 1999)* the study says that one of the most fundamental and elementary problems in finance is to estimate the present value of a future cash flow. If the discount rate (interest rate) is constant and equal to 'r', then the present value of a future cash flow $C(t)$, where $C(t)$ is a function of time is 't' is given by:

$$p(t) = \sum_{t=1}^{T} \frac{C(t)}{(1+r)^t} = \sum_{t=1}^{T} e^{-rt} C(t). \tag{1.2}$$

assuming continuous compounding and present value function, *p*(*t*) is a function of '*t*'. The time is bounded between '0'and some finite quantity '*T*'. In the limiting case, the summation is replaced by an integral and the above present value equation can be expressed as:

$$p(r) = \int_0^T e^{-rt} C(t)\,dt. \tag{1.3}$$

Due to the presence of the integral, the domain of the computation changes from time '*t*' to rate '*r*' and thus, the present value becomes a function of the rate '*r*'.

The finite Laplace transform of a continuous function *f* (*t*) in (0, *T)* is defined by

$$\mathrm{L}_T\left\{f(t)\right\} = \tilde{f}(s,T) = \int_0^T e^{-st} f(t)\,dt, \tag{1.4}$$

where 's' is a real or complex number and T is a finite number that may be positive or negative so that (1.4) can be defined in any interval#- T_1, T_2) (Buschman, 1991) in Finite Laplace Transform. The idea of introducing the finite Laplace transform in 0#£## £ T in order to extend the power and usefulness of the usual Laplace transform in 0#£##£¥ was shown by Lokenath Debnath & Dambaru Bhatta, (2015). Recent developments were carried out by Lakshmi Gorty and Khinal Parmar where finite Laplace transform of spherical Bessel Function is developed. The finite Laplace transform was mainly developed by Debnath et.al, (2015). Finite Laplace transform was used to solve initial value problems and boundary value problems (Richard Datko, 1980). Finite Laplace transform was used to solve partial differential equation problems (Buschman, 1991). Richard Datko, (1980) illustrated applications of the finite Laplace transform to linear control problems. Goldwyn, Sriram & Graham, (1967) showed time optimal control of a linear diffusion process in applications in the Engineering field. The author (Agnė Pivorienė, 2017) investigated the feasibility of real options approach and traditional DCF analysis for assessment of strategic investment projects under environmental uncertainty.

Literature review of continuous finance

The author (David, 2001) states that when an entity uses the interest method, the statement requires a careful description of: the cash flows to be used (promised cash flows, expected cash flows or some other estimate), the convention governing the choice of an interest method (effective rate or some other rate), how the rate is applied (constant effective rate or a series of annual rates) and how the entity will report changes in the amount or timing of estimated cash flows. Agnė Pivorienė, (2017) analysed the valuation of financial instruments; project valuation techniques usually assume that expected cash flows are discounted at discrete intervals, e.g., daily, monthly, quarterly, semi-annually, or annually. In some instances, especially for high-risk investments, continuous discounting can be used for more precise valuation. The author also mentioned an example of the technique of continuous discounting which is widely used in financial options valuation and primarily in the Black-Scholes option pricing model. In another study, a discrete-continuous project scheduling problem with discounted cash flows was considered. In discrete-continuous project scheduling, activities required for processing discrete and continuous resources were also analysed. The processing rate of an activity is the same function of the amount of the continuous resource allotted to this activity at a time (Grzegorz Waligóra, 2015).

Preliminary Results

Let $k \in \mathrm{R}$ be the real free parameter and the fractional-Laplace transform is defined with kernel of the transform and also known as the generalised Lorentzian

$$K(s,t)=\frac{1}{\left(1+\frac{st}{k}\right)^{k+r}}$$

(Treumann & Baumjohann, 2014).

$$L_k\left[f(t);r\right]=P(r)=g_k(s,r)=\int_0^{\infty}\frac{f(t)}{\left(1+\frac{st}{k}\right)^{k+r}}dt. \tag{3.1}$$

Inverse transform is written as:

$$L_k^{-1}\left[g_k(s,r)\right]=f(t)=\int_{c-i\infty}^{c+i\infty}\frac{g_k(s)}{\left(1+\frac{st}{k}\right)^{-k-r}}dt. \tag{3.2}$$

and convention that the index on the transform L_k refers to k in the exponent. These forms are obtained by replacing the exponentials in the ordinary Laplace transform and its inverse by generalised Lorentzian $\left(1+\frac{ist}{k}\right)^{-(k+r)}$. The philosophy in mind is that for any rational number $k, r \in$ □, the generalised Lorentzians, for $k \to \infty$ and r fixed, asymptotically approach exponentials $e^{\mp st}$ (Bell, 1968).

The equation (3.1) is now an approximation of equation (1.1), where the r (rate) domain acts like the Laplace domain s. Therefore, from Buser, (1986) the present value of a future cash flow $P(r)$ is the Laplace transform of the cash flow and future cash flow $C(t)$, where $C(t)$ a function of time is $'t'$ can be given by:

$$p(t)=\sum_{t=1}^{T}\left(1\pm\frac{st}{k}\right)^{-k-r}C(t). \quad (3.3)$$

Of course, one needs to be cognizant of the fact that the bounds of the integral above are from 0 to infinity. In other words, the Laplace transform equation (1.3) exactly translates into the discrete time present value equation (1.2) only when we are considering a very long period of time.

For 100 years, if a single investment of Rs.100 is invested at an annual rate of interest of 6.5%, then the value after 100 years becomes Rs.54,320. Using Laplace transforms using function as $f(t)=8t^2$, and $'s'$ as a parameter and the value is considered as $s=6.5/100$ and $'T'$ in (1.4) is a 100-year period (100 years is long enough to be the real life equivalent of infinity as in (1.1)), then Laplace transforms of the single investment is obtained as Rs.55,754. The MATLAB program for this calculation is represented as:

subs(int(exp(-s*t)*2*(2*t)^2,'t',0,100),'s',6.5/100). (3.4)

Considering the formula as in (3.3), the value of Laplace transforms is given by Rs.55,754.

The error considering (3.3) with respect to (1.4) is given as:

abs(100*(1+6.5/100)^100-subs(int(exp(-s*t)*2*(2*t)^2,'t',0,100),'s',6.5/100))=1433.80. (3.5)

abs(100*(1+6.5/2.06)^(-2.06+6.5)-subs(int(exp(-s*t)*2*(2*t)^2,'t',0,100),'s',6.5/100))=43.3470 (3.6)

Not many analytic solutions exist for present value problems, but k -Laplace transforms used in this study is more efficient in estimating than developed in Latte, (2010). The observation made here (3.6) is more effective and efficient than (3.5). Hence, the k -Laplace transforms developed in Treumann & Baumjohann, (2014) can be used in future for estimating the time value of money.

Managerial applications

As mentioned in (CPA guys, 2017): 'Valuing Small Businesses, overcoming accounting and finance hurdles for new business owners' explains one of the methods for valuing a business i.e. the “Income Approach”:
The “Income Approach” (often referred to as the Multiplier Method)
Step 1: Find the SDI (Seller's Discretionary Income).
Step 2: Multiply the SDI by the “right” multiplier.
Step 3: Increase or decrease the price by the value of “Working Capital”.
Current Assets – Current Liabilities = Working Capital.

The multiplier will be a function of the industry and the risk involved with owning the particular business being valued. Certain industries may have more growth potential than others, and sell for higher multipliers. After ascertaining the industry-specific multiplier, looking into other factors that positively or negatively affect the risk of owning the business is a question. Factors such as location and customer loyalty come to mind. These factors can be used to adjust the multiplier.

As in equation (3.4), the multiplier has been chosen as an example. Depending on the factors of the industry and the risk involved with owning the particular business and its growth potential, (3.4) can be suggested, so that appropriate analysis can be done.

Limitations of the methodology

1. If the principal amount is not deposited in time, the effective interest rate would be the original rate as on the date of the start.
2. If the amount is deposited on variable rate, then it may be effected at the time of each calculation, and may not hold for

continuous finance equation (3.4).

3. The general principles only depend on amount of future cash flows and their timing.
4. The idea of what would be the appropriate time to start depositing with appropriate rate of interest and the optimised benefits may not be forecasted every time.
5. (3.4) represents a life of 0 to 100 years, but the investments or depositing may not be at that early stage. And end age is again not predictable.
6. The representation (3.4) is limited to the condition that the savings/ investments are continuous for a certain time period. If there is any discontinuity, then (3.4) will not hold true.

Conclusion

In the present study,*k*-Laplace transforms developed is more efficient than the existing Laplace transforms application to estimate the time value of money. One of the most fundamental and elementary problems in finance is to estimate the present value of a future cash flow, which is shown by using -Laplace transforms.

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